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THREE-DIMENSIONAL DIFFUSIVE BOUNDARY-LAYER PROBLEMS

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1. PROBLEM FORMULATION. CHOICE OF COORDINATE SYSTEM

Consider three-dimensional viscous incompressible laminar flow past a solid or liquid particle of arbitrary shape with convective diffusion to the surface. It is assumed that the Peclet number $\mathrm{Pe} = aUD^{-1}$ is large; here a is the characteristic dimension of the particle (it is usually the radius of an equivalent sphere by volume), U is the characteristic flow velocity (at infinity), D is the diffusion coefficient. It is also assumed that concentration C_* is constant at the surface and away from it, equal to C_s and C_∞, respectively, and the flow field is determined from the solution of the corresponding hydrodynamic problem of the flow past the particle.

Orthogonal curvilinear coordinate system in ξ, η, λ connected to the body surface and streamlines is used in the analysis as in [1, 2]. The directions of unit vectors at any point M in the surrounding fluid are given by e_ξ, e_η, e_λ (Fig. 1). The unit vector e_ξ is determined by the direction of the normal to surface of the particle passing through the point M; the unit vector e_η is given by the direction of the projection of the velocity vector at the point M on the plane perpendicular to e_ξ; the unit vector e_λ is chosen such that the system of unit vectors e_ξ, e_η, e_λ is a right-handed orthogonal triad (Fig. 1). The origin of the coordinate system and the procedure for computing curvilinear coordinates (i.e., the dependence of metric tensor components $g_{\xi\xi}$, $g_{\eta\eta}$, $g_{\lambda\lambda}$ on ξ, η, λ), are chosen from the point of view of convenience in each particular case; for concreteness, we further assume that the surface of the particle is given by a fixed value $\xi = 0$. In such a coordinate system the fluid velocity vector at each point is given by $v = \{v_\xi, v_\eta, 0\}$.

The equation of continuity for an incompressible fluid has the form

$$\operatorname{div}\mathbf{v} = \frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial\xi}\left(v_\xi\sqrt{\frac{g}{g_{\xi\xi}}}\right) + \frac{\partial}{\partial\eta}\left(v_\eta\sqrt{\frac{g}{g_{\eta\eta}}}\right)\right] = 0. \tag{1.1}$$

The function $\psi(\xi, \eta, \lambda)$ is determined as the solution to the system

$$\frac{\partial\psi}{\partial\xi} = v_\eta\sqrt{\frac{g}{g_{\eta\eta}}},\quad \frac{\partial\psi}{\partial\eta} = -v_\xi\sqrt{\frac{g}{g_{\xi\xi}}}. \tag{1.2}$$

Then the equation of continuity (1.1), which coincides with the condition for integrability of the system (1.2), is automatically satisfied. The constant of the integration in Eq. (1.2) is chosen such that the function ψ becomes zero at the surface.

The surface $\psi(\xi, \eta, \lambda) = \mathrm{const}$ wholly consists of streamlines. The function ψ has a simple physical meaning: It is the three-dimensional analog of stream function. In the plane and axisymmetric cases ψ coincides with stream function.

In nondimensional variables the equation of stationary convective diffusion and boundary conditions in curvilinear coordinate system ξ, η, λ are written in the following form using Eq. (1.2):

$$-\frac{\partial(c,\psi)}{\partial(\xi,\eta)} = \frac{1}{\mathrm{Pe}}\left\{\frac{\partial}{\partial\xi}\left(\frac{\sqrt{g}}{g_{\xi\xi}}\frac{\partial c}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\frac{\sqrt{g}}{g_{\eta\eta}}\frac{\partial c}{\partial\eta}\right) + \frac{\partial}{\partial\lambda}\left(\frac{\sqrt{g}}{g_{\lambda\lambda}}\frac{\partial c}{\partial\lambda}\right)\right\}; \tag{1.3}$$

$$\xi = 0,\ c = 0;\ \xi \to \infty,\ c \to 1, \tag{1.4}$$

$$c = (C_* - C_s)/(C_\infty - C_s),\quad \mathrm{Pe} = aU/D,\quad g = g_{\xi\xi}g_{\eta\eta}g_{\lambda\lambda},$$

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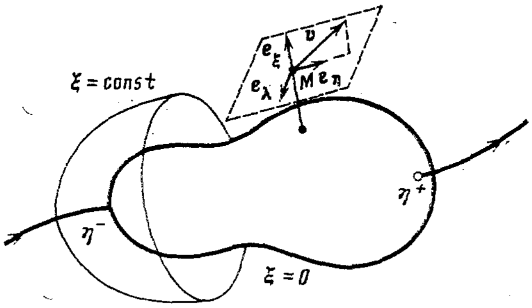

Fig. 1

where the Jacobian of functions c and ψ is given on the left-hand side of the equation.

Consider first the case when there are only two isolated singularities on the surface of the particle (cases when there are no stagnation points on the surface and there is a bound vortex behind the particle are not considered in this work). Further, the stagnation point on the surface of the particle will be called the inflow (outflow) point, if in its neighborhood the normal velocity component of the fluid is towards (away from) the surface of the body; diffusive boundary layer located near the surface of the particle is "generated" in the neighborhood of the inflow points. Let the inflow point be denoted by the coordinate η^- and the outflow point by η^+ (see Fig. 1).

The no-slip condition must be satisfied at the solid (liquid) surface in viscous flow and hence in the case of incompressible flow the three-dimensional analog for stream function can be represented in the following form using Eq. (1.2)

$$\xi \to 0,\ \psi \to \xi^n f(\eta, \lambda). \tag{1.5}$$

In certain problems with laminar viscous flow past smooth solid particles, the parameter n usually takes the value two. Nevertheless, there are some examples of Stokes flow in which n = 3 [3]. In the case of a droplet or a bubble and also in the case of incompressible flow past the particle, n = 1.

2. METHOD OF SOLUTION. FUNDAMENTAL EQUATIONS FOR NONDIMENSIONAL INTEGRAL FLOW

Assuming that the region near the particle is characterized by the inequalities $\xi \geqslant 0$, $\psi \geqslant 0$, we switch from older to newer (stretched) coordinates in the diffusive boundary layer:

$$(\xi, \eta, \lambda) \to (Y, \eta, \lambda),\ Y = \mathrm{Pe}^{\nu}\xi,\ \nu = (n+1)^{-1}.$$

Eliminating higher order terms of the series with the small parameter $\mathrm{Pe}^{-\nu}$ from Eq. (1.3) using Eq. (1.5) (here, it so happens that the last two terms within the curved brackets can be neglected compared to the first), we obtain the diffusion boundary-layer equations in the following form:

$$-\frac{\partial(c, Y^n f)}{\partial(Y, \eta)} = \frac{\sqrt{g^0}}{g^0_{\xi\xi}}\frac{\partial^2 c}{\partial Y^2}, \tag{2.1}$$

which parametrically depends on the third curvilinear coordinate λ, analogous to the cyclical variables in analytical mechanics; here and in what follows, index zero denotes quantities at the body surface with $\xi = 0$.

Using the new variables

$$\zeta = \mathrm{Pe}^{\nu}\xi f^{\frac{1}{n}} = \mathrm{Pe}^{\nu}\psi^{\frac{1}{n}},\quad t = t(\eta, \eta^-; \lambda) = \frac{1}{n}\int\limits_{\eta^-}^{\eta}\frac{\sqrt{g^0}}{g^0_{\xi\xi}} f^{\frac{1}{n}} d\eta, \tag{2.2}$$

the boundary-layer problem on diffusion (2.1) and (1.4) is reduced to the following boundary-value problem:

$$\frac{\partial c}{\partial t} - \zeta^{1-n}\frac{\partial^2 c}{\partial \zeta^2} = 0,\quad c(\zeta, 0) = 1,\quad c(0, t) = 0,\quad c(\infty, t) = 1. \tag{2.3}$$

Its solution has the form

$$c=\frac{1}{\Gamma(\nu)}\gamma\left(\nu,\nu^2\frac{\zeta^{n+1}}{t}\right),\quad \nu=\frac{1}{n+1},\quad \gamma(\nu,x)=\int_0^x x^{\nu-1}e^{-x}dx,\quad \Gamma(\nu)=\gamma(\nu,+\infty). \tag{2.4}$$

Nondimensional local diffusion flow is determined by the normal derivative of the concentration (2.4) at the surface of the body

$$j(\eta,\lambda)=\frac{1}{\sqrt{g^0_{\xi\xi}}}\left(\frac{\partial c}{\partial\xi}\right)_{\xi=0}=\mp\nu^{(1-n)\nu}\frac{\mathrm{Pe}^\nu|f(\eta,\lambda)|^{\frac{1}{n}}}{\Gamma(\nu)\sqrt{g^0_{\xi\xi}(\eta,\lambda)}}t^{-\nu}(\eta,\eta^-;\lambda). \tag{2.5}$$

For the complete nondimensional diffusive flow dissolved in a liquid substance at the surface of the particle $S=\{\xi=0,\ \eta^-\leqslant\eta\leqslant\eta^+,\ 0\leqslant\lambda\leqslant\Lambda\}$ we have

$$I=\iint_S jdS=\int_0^\Lambda\int_{\eta^-}^{\eta^+} j\,(g^0_{\eta\eta}g^0_{\lambda\lambda})^{\frac{1}{2}}d\eta d\lambda=\nu^{-2n\nu}\frac{\mathrm{Pe}^\nu}{\Gamma(\nu)}\int_0^\Lambda t^{n\nu}(\eta^+,\eta^-;\lambda)\,d\lambda,\quad \nu=\frac{1}{n+1}. \tag{2.6}$$

It is worth noting that a more complex analysis of three-dimensional boundary layer was carried out in [4, 5], where a transformation leading to the separation of variables was found for the stationary convective diffusion equations written in the usual boundary-layer coordinates which are frequently used in similar hydrodynamic boundary layer problems and connected only to the surface of the body (and not to the streamlines). A local orthogonal curvilinear coordinate system similar to ξ, η, λ was used in the boundary layer in [6] for the case of an arbitrary three-dimensional flow past a solid particle which corresponds to a value $n = 2$ in Eq. (1.5). Equations for the concentration distribution in diffusive boundary layer for the two-dimensional (plane or axisymmetric) flow past droplets and particles of arbitrary shape, described in one or the other notations in different coordinate systems, have appeared in many works (see e.g., [3, 7-10]).

The case $\psi\leqslant 0$ and the case where the flow region around the particle is given by the inequality $\xi\leqslant 0$ are considered in a similar manner. In the general case, when there are more than two stagnation points and lines on the surface of the particle (i.e., the analog of stream function changes sign), the flow region near the surface of the body is divided into segments in each of which the sign of the stream function analog is constant. It is possible to show that the concentration distribution and local diffusive flow at each of these segments are determined by Eqs. (2.4) and (2.5), where

$$\zeta=\mathrm{Pe}^\nu|\psi|^{\frac{1}{n}},\quad t=t(\eta,\eta_k^-;\lambda)=\frac{1}{n}\left|\int_{\eta_k^-}^{\eta}\frac{\sqrt{g^0}}{g^0_{\xi\xi}}|f|^{\frac{1}{n}}d\eta\right|, \tag{2.7}$$

and η_k^- ($k = 1, \ldots, K$) is the stagnation line (point) of the inflow located at the boundary (inside) of the segment. Complete integral flow is computed by summing up integral flows in all segments.

In the two-dimensional case ($\partial/\partial\lambda = 0$), Eqs. (2.4)-(2.7) reduce to results of [3].

In order to use Eqs. (2.4)-(2.7) in the general case of three-dimensional flow, it is necessary to solve auxiliary problems on the determination of curvilinear coordinate system ξ, η, λ and find an expansion for the analog stream function near the surface of the droplets and particles (1.5). As a rule, the initial information on the flow field makes it possible to obtain directly only the distribution of liquid velocities near the particle in certain orthogonal fixed system of coordinates ξ, μ, χ fixed only to its surface $\xi = 0$ (and not connected to streamlines). In view of this it is often necessary to compute the quantities f and t present in Eqs. (2.4)-(2.6) directly in the initial orthogonal coordinate system ξ, μ, χ. Consequently, we will now show how, using the asymptote of tangential vector component of the liquid flow velocity v_τ near the particle surface

$$\xi\to 0,\ \mathbf{v}_\tau\to\xi^{n-1}(F_\mu\mathbf{e}_\mu+F_\chi\mathbf{e}_\chi),\ F_\mu=F_\mu(\mu,\chi),\ F_\chi=F_\chi(\mu,\chi), \tag{2.8}$$

to obtain functions (1.5) and (2.7) in the coordinate system ξ, μ, χ.

Comparing Eqs. (1.2), (1.5), and (2.8) we get the following equations for functions f and t:

$$f = \frac{1}{n} F \frac{\sqrt{g}}{\sqrt{g_{\eta\eta}}}, \quad t = n^{-\frac{n+1}{n}} \left| \int\limits_{\eta^-}^{\eta} g_{\xi\xi}^{\frac{1-n}{2n}} g_{\lambda\lambda}^{\frac{1+n}{2n}} F^{\frac{1}{n}} g_{\eta\eta}^{\frac{1}{2}} d\eta \right|,$$
$$F = \left(F_\mu^2 + F_\chi^2\right)^{1/2}. \tag{2.9}$$

Here and in what follows the index zero on all the above-mentioned components of the metric tensor is omitted (it should be remembered that all these quantities are taken at the particle surface with $\xi = 0$).

Considering that the orthogonal curvilinear coordinates $\eta = \eta(\mu, \chi)$ and $\lambda = \lambda(\mu, \chi)$ should satisfy the conditions $\mathbf{v}_\tau = \text{const}\, \nabla\eta$ and $(\mathbf{v}_\tau \cdot \nabla\lambda) = 0$, the following equations are obtained for the determination of η, λ:

$$\frac{1}{F_\mu \sqrt{g_{\mu\mu}}} \frac{\partial\eta}{\partial\mu} - \frac{1}{F_\chi \sqrt{g_{\chi\chi}}} \frac{\partial\eta}{\partial\chi} = 0; \tag{2.10}$$

$$\frac{F_\mu}{\sqrt{g_{\mu\mu}}} \frac{\partial\lambda}{\partial\mu} + \frac{F_\chi}{\sqrt{g_{\chi\chi}}} \frac{\partial\lambda}{\partial\chi} = 0. \tag{2.11}$$

Since the square of the surface length is invariant in the transformation from the old μ, χ to the new η, λ coordinate system:

$$g_{\mu\mu} d\mu^2 + g_{\chi\chi} d\chi^2 = g_{\eta\eta} d\eta^2 + g_{\lambda\lambda} d\lambda^2$$
$$\left(d\eta = \frac{\partial\eta}{\partial\mu} d\mu + \frac{\partial\eta}{\partial\chi} d\chi,\ d\lambda = \frac{\partial\lambda}{\partial\mu} d\mu + \frac{\partial\lambda}{\partial\chi} d\chi\right). \tag{2.12}$$

By a few simplifications of Eq. (2.12) using Eqs. (2.10) and (2.11), it is possible to obtain an expression for the metric coefficient $g_{\lambda\lambda}$:

$$g_{\lambda\lambda} = g_{\mu\mu} \frac{F_\chi^2}{F^2} \left(\frac{\partial\lambda}{\partial\mu}\right)^{-2} = g_{\chi\chi} \frac{F_\mu^2}{F^2} \left(\frac{\partial\lambda}{\partial\chi}\right)^{-2}. \tag{2.13}$$

Similar equations for the other metric coefficient $g_{\eta\eta}$ are obtained from Eq. (2.13) by replacing λ by η, μ by χ, and χ by μ.

In the integrand of the expression for t (2.9) the integration variable η is changed to the variable μ, using the following relation which is valid when λ = const:

$$\sqrt{g_{\eta\eta}} d\eta = \sqrt{g_{\mu\mu}} \frac{F}{F_\mu} d\mu = \sqrt{g_{\chi\chi}} \frac{F}{F_\chi} d\chi, \tag{2.14}$$

which is obtained from (2.12) in view of the fact that the equality λ = const is fulfilled on the integral curves of the characteristic differential equations

$$\frac{\sqrt{g_{\mu\mu}}}{F_\mu} d\mu = \frac{\sqrt{g_{\chi\chi}}}{F_\chi} d\chi, \tag{2.15}$$

corresponding to the partial differential equation (2.11). As a result we obtain

$$t = t(\mu, \mu^-; \lambda) = n^{-\frac{n+1}{n}} \left| \int\limits_{\mu^-}^{\mu} \left\{ g_{\xi\xi}^{\frac{1-n}{2n}} g_{\mu\mu}^{\frac{1+2n}{2n}} \frac{|F_\chi|^{\frac{1+n}{n}}}{F_\mu} \left(\frac{\partial\lambda}{\partial\mu}\right)^{-\frac{n+1}{n}} \right\}_\lambda d\mu \right| =$$
$$= n^{-\frac{n+1}{n}} \left| \int\limits_{\mu^-}^{\mu} \left\{ g_{\xi\xi}^{\frac{1-n}{2n}} g_{\mu\mu}^{\frac{1}{2}} g_{\chi\chi}^{\frac{1+n}{2n}} |F_\mu|^{\frac{1}{n}} \left(\frac{\partial\lambda}{\partial\chi}\right)^{-\frac{n+1}{n}} \right\}_\lambda d\mu \right|. \tag{2.16}$$

Here and in what follows, the subscript λ after the bracket ($\}$) denotes that the respective quantity is obtained at λ = const. The transformation within the integrand (2.9) from η to another variable is carried out by exchanging the local coordinates $\mu \rightleftarrows \chi$ in (2.16).

In order to study the mass transfer to a spherical particle or droplet (bubble) in an arbitrary three-dimensional laminar flow, it is useful to have equations to determine the function t in spherical coordinate system r, θ, φ, fixed to the center of the particle. In this case we have (metric coefficients are obtained at the particle surface)

$$\xi = r - 1,\ \mu = \theta,\ \chi = \varphi;\ g_{\xi\xi} = 1,\ g_{\mu\mu} = 1,\ g_{\chi\chi} = \sin^2\theta. \tag{2.17}$$

Using Eq. (2.16) and Eq. (2.17) for the variable t, the following equations are obtained:

$$\frac{t}{n^\sigma} = \left| \int_{\theta^-}^{\theta} \left\{ \sin^\sigma\theta\, |F_\theta|^{\frac{1}{n}} \left(\frac{\partial\lambda}{\partial\varphi}\right)^{-\sigma} \right\}_\lambda d\theta \right| = \left| \int_{\theta^-}^{\theta} \left\{ \frac{|F_\varphi|^\sigma}{F_\theta} \left(\frac{\partial\lambda}{\partial\theta}\right)^{-\sigma} \right\}_\lambda d\theta \right| =$$
$$= \left| \int_{\varphi^-}^{\varphi} \left\{ \sin^{\sigma+1}\theta\, \frac{|F_\theta|^\sigma}{F_\varphi} \left(\frac{\partial\lambda}{\partial\varphi}\right)^{-\sigma} \right\}_\lambda d\varphi \right| = \left| \int_{\varphi^-}^{\varphi} \left\{ \sin\theta\, |F_\varphi|^{\frac{1}{n}} \left(\frac{\partial\lambda}{\partial\vartheta}\right)^{-\sigma} \right\}_\lambda d\varphi \right|,\ \sigma = \frac{n+1}{n}. \tag{2.18}$$

Thus, the procedure for the computation of integral flow (or the mean Sherwood number) at the particle surface is carried out sequentially in four stages: The tangential component of the velocity near the particle surface (2.8) is first determined; then the general solution of the characteristic equation (2.15) is found and by replacing the constant of integration in it by λ, the relation $\lambda = \lambda(\mu, \chi)$ is obtained; in the third stage the variable t is computed from any of Eqs. (2.16) and (2.18) (which are chosen from the point of view of convenience) taking into consideration that the integrand should be initially expressed only through the coordinate λ and the variable with respect to which the integration is carried out; in the final step the integrals similar to (2.6) are computed.

The use of Eqs. (2.16) and (2.18) is described with the help of a few examples.

3. MASS TRANSFER TO SPHERICAL DROPLET OR SOLID PARTICLE IN AN ARBITRARY IN PURE SHEAR FLOW

Consider mass transfer to a spherical droplet or a solid particle in an arbitrary, pure, linear shear flow whose velocity distribution at infinity has the form

$$r \to \infty,\ v_i = E_{ij}x_j + o(1),\ E_{ij}\delta_{ij} = 0,\ E_{ij} = E_{ji}, \tag{3.1}$$

where v_i and E_{ij} are nondimensional components of velocity and the shear stress tensor (appropriate values are chosen in each concrete case while normalizing these) described in Cartesian coordinates whose origin is fixed to the center of the particle; here and in what follows, repetition of indices i and j indicates summation (i, j = 1, 2, 3); δ_{ij} is the Kronecker delta. The sum of the diagonal elements equals zero if the flow is incompressible $(\mathbf{v}\cdot\nabla) = 0$; symmetry of shear stress tensor when the indices i and j are interchanged indicates the absence of rotation at infinity (in other words, purely deformed flow is being considered).

The solution to the hydrodynamic problem of an arbitrary purely deformed Stokes' flow past a spherical droplet (3.1) has the form [11]

$$v_i = E_{ij}x_j\left(1 - \frac{\beta}{\beta+1}\frac{1}{r^5}\right) - \frac{5}{2}E_{jk}x_ix_jx_k\left(\frac{\beta+2/5}{\beta+1}\frac{1}{r^5} - \frac{\beta}{\beta+1}\frac{1}{r^7}\right),$$
$$r = \left(x_1^2 + x_2^2 + x_3^2\right)^{1/2}, \tag{3.2}$$

where β is the ratio of the viscosities of the droplet and the surrounding medium; the value $\beta = \infty$ corresponds to a solid particle and $\beta = 0$ denotes gas bubble.

The symmetric tensor **E** can be reduced to the diagonal form by the transformation of the coordinate system with components E_m (m = 1, 2, 3), which are determined by solving the cubic equation $\det\|E_{ij} - E_m\delta_{ij}\| = 0$. Diagonal elements E_1, E_2, E_3 reduced to the principal axes of the tensor **E** determine the strength of the stretched (compressed) flow along the coordinate axis. The symmetric shear tensor has three invariant scalars:

$$J_1 = E_{ij}\delta_{ij} = E_1 + E_2 + E_3 = 0,\ J_2 = (E_{ij}E_{ij})^{1/2} = \left(E_1^2 + E_2^2 + E_3^2\right)^{1/2},$$
$$J_3 = |\det\|E_{ij}\||^{1/3} = |E_1E_2E_3|^{1/3}, \tag{3.3}$$

which remain unchanged with any transformation (with reflections) of the original coordinate system. In accordance with incompressibility condition $J_1 = 0$ only two of the three diagonal elements will be independent.

Further, the Cartesian coordinate system fixed to the principal shear is represented by X_1, X_2, X_3 and without loss of generality, assume that $E_1 \geqslant E_2 \geqslant 0$, $E_3 < 0$ (i.e., $|E_3| = \max_m |E_m|$)

The spherical coordinate system fixed to the principal axes of the shear tensor, the higher order terms in the expansion for tangential velocity components (3.2) near the droplet surface, and the solid particle are given by (as $r \to 1$)

$$v_\theta = (1/2)q(n)\xi^{n-1}\sin 2\theta\{-3E_3 + (E_1 - E_2)\cos 2\varphi\},$$
$$v_\varphi = q(n)\xi^{n-1}\sin\theta\sin 2\varphi(E_2 - E_1),$$
$$q(1) = 1/(2(\beta + 1)),\ q(2) = 5/2,\ \xi = r - 1, \tag{3.4}$$

where $n = 1$ corresponds to a droplet with moderate viscosity ($\beta \leq O(1)$) and $n = 2$ indicates solid particle ($\beta = \infty$).

It is seen from Eq. (3.4) that the flow field is three dimensional when $E_1 \neq E_2$. It should be mentioned that axisymmetric case, when $E_1 = E_2 = -E_2/2$, was considered in [12]. Mass transfer in plane shear flow past solid sphere given by parameters $E_1 = -E_3$, $E_2 = 0$; $n = 2$ in Eq. (3.4) (three-dimensional problem) was studied in [6, 13].

It follows from (3.4) that there are six isolated singularities on the surface of the spherical droplet or the solid particle located along the principal axes of the shear stress tensor: 1) $\theta = 0$; 2) $\theta = \pi$; 3) $\theta = \pi/2$, $\varphi = 0$; 4) $\theta = \pi/2$, $\varphi = \pi$; 5) $\theta = \pi/2$, $\varphi = \pi/2$; 6) $\theta = \pi/2$, $\varphi = 3\pi/2$. The first two are the leading edge (inflow) stagnation points and the following two are the trailing edge (outflow) stagnation points and the last two are neutral points (saddle point type singularity). In the limiting axisymmetric case $E_1 = E_2$ the last four isolated singularities at the equator of the droplet $\theta = \pi/2$ are replaced by the appearance of trailing edge (outflow) stagnation line.

The characteristic equation determining the relation between the initial curvilinear coordinates λ and spherical coordinates θ and φ is the same for the case of the droplet and the solid sphere and has the form

$$\frac{2d\theta}{\sin 2\theta} = \frac{-3E_3 + (E_2 - E_1)\cos 2\varphi}{\sin 2\varphi\,(E_1 - E_2)}\,d\varphi. \tag{3.5}$$

Equations (2.15), (2.17), and (3.4) have been used in deriving these equations.

It is not difficult to show that the general solution of Eq. (3.5) can be expressed in the form

$$C = \mathrm{tg}^2\,\theta\,\mathrm{tg}^{\varkappa}\,\varphi\,\sin 2\varphi,\ \varkappa = 3(E_1 + E_2)/(E_1 - E_2), \tag{3.6}$$

where C is an arbitrary constant; the component E_3 was eliminated using the equation $J_1 = 0$ (3.3) to obtain the index χ. The relation between the curvilinear coordinate λ and spherical coordinates is obtained by assuming in Eq. (3.6) that $C \equiv \lambda$ (obviously, it is also possible to assume $\lambda = \Phi(C)$, where Φ is any arbitrary sufficiently "good" function).

The qualitative behavior of the limiting streamlines on the surface of the spherical droplet or solid particle in the first quadrant $0 \leqslant \theta,\ \varphi \leqslant \pi/2$ is shown in Fig. 2; the value of the variable $\lambda = C$ (3.6) varies from zero to infinity. The integral diffusion flow in this segment of the surface is computed by using Eq. (2.6) at $\Lambda = +\infty$. The integrand in Eq. (2.6) is determined by using the last equation in (2.18), taking into account the relation $\partial\lambda/\partial\theta = 4\lambda/\sin 2\theta$ (which results from Eq. (3.6) when $C = \lambda$) and also in view of the fact that the coordinate in the given region varies within the limits $0 \leqslant \varphi \leqslant \pi/2$. For the function

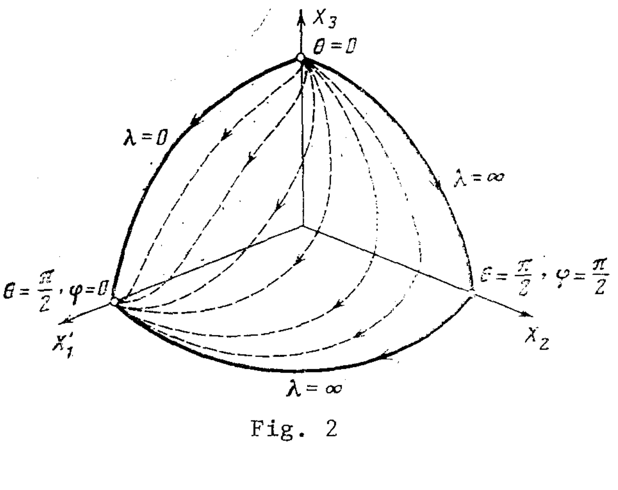

Fig. 2

$t(\eta^+, \eta^-; \lambda)$ we get

$$t(\eta^+, \eta^-; \lambda) = \frac{[\,|E_1 - E_2|\, q(n)]^{\frac{1}{n}}}{(4n\lambda)^{\frac{n+1}{n}}} \int\limits_0^{\frac{\pi}{2}} \{\sin\theta \sin 2\theta\}_\lambda^{\frac{n+1}{n}} \sin^{\frac{1}{n}} 2\varphi d\varphi =$$

$$= \frac{1}{4} \frac{[\,|E_1 - E_2|\, q(n)]^{\frac{1}{n}}}{(2n)^{\frac{n+1}{n}}} \int\limits_0^1 \frac{z^{\frac{\varkappa n+\varkappa-n+3}{4n}} (1-z)^{\frac{\varkappa n+\varkappa-2n}{2n}}}{\left[z^{\frac{\varkappa+1}{2}} + \frac{\lambda}{2}(1-z)^{\frac{\varkappa-1}{2}}\right]^{\frac{3}{2}\frac{n+1}{n}}} dz. \tag{3.7}$$

In the case of the droplet in a plane shear flow (three-dimensional flow field) described by the parameters $n = 1$, $\varkappa = 3$ ($E_1 = -E_3$, $E_2 = 0$), the integral (3.7) can be expressed in terms of elementary functions

$$t(\eta^+, \eta^-; \lambda) = \frac{|E_1|}{32(\beta+1)} \int\limits_0^1 \frac{z^2(1-z)^2 dz}{\left[z^2 + \frac{\lambda}{2}(1-z)\right]^3} = \tag{3.8}$$

$$= \frac{|E_1|}{64(\beta+1) z_2^3} \left(\frac{d^2}{dY^2} Y^2 \frac{d^2}{dY^2} \frac{\ln Y}{Y-1}\right)_{Y=z_1/z_2}, \quad z_{1,2} = \frac{\lambda}{4} \pm \sqrt{\frac{\lambda^2}{16} - \frac{\lambda}{2}}.$$

Equations (5) and (4) in [14] [pp. 225 and 82 in the Russian translation] have been used in deriving Eq. (3.8); z_1 and z_2 correspond to the roots of the denominator in the integrand of Eq. (3.8); when $0 \leqslant \lambda \leqslant 8$ the roots $z_{1,2}$ are complex and after differentiation with respect to Y Eq. (3.8) should be rewritten, taking into consideration the relation $\ln Y = 2i \arctan \sqrt{8\lambda^{-1} - 1}$.

The computation of the integral diffusion flow over the entire droplet surface is carried out using Eqs. (2.6) and (3.8) with $\Lambda = \infty$, taking into account that the integral flow in the given surface region (see Fig. 2) comprises 1/8 of the whole. Computations for the mean Sherwood number $\mathrm{Sh} = I(4\pi)^{-1}$ give

$$\mathrm{Sh} = 0.737(\beta+1)^{-1/2}\mathrm{Pe}^{1/2}, \quad \mathrm{Pe} = a^2|E_1^*|D^{-1}(E_1^* = -E_3^*,\ E_2^* = 0). \tag{3.9}$$

The asterisk denotes dimensional shear stress components.

Choosing the second dimensional invariant in (3.3) $J_2^* = \sqrt{2}|E_1^*|$ as the characteristic length scale for shear it is possible to rewrite Eq. (3.9) in the following equivalent form:

$$\mathrm{Sh} = 0.620(\beta+1)^{-1/2}\mathrm{Pe}_M^{1/2}, \quad \mathrm{Pe}_M = a^2 J_2^* D^{-1}, \quad J_2^* = (E_{ij}^* E_{ij}^*)^{1/2}, \tag{3.10}$$

where Pe_M is the modified Peclet number.

Results [12] obtained for the axisymmetric shear ($E_1^* = E_2^* = -E_3^*/2$, $J_2^* = \sqrt{6}|E_1^*|$) and written in terms of modified Peclet number are given by

$$\mathrm{Sh} = 0.624(\beta+1)^{-1/2}\mathrm{Pe}_M^{1/2}, \quad \mathrm{Pe}_M = a^2 J_2^* D^{-1}. \tag{3.11}$$

If, in the general case of arbitrary pure shear flow (3.1), the characteristic length scale is taken as J_2^*, that corresponds to the value $J_2 = 1$ in (3.3), then the mean Sherwood number, which is a scalar quantity, should depend only on the ratio of the dimensional invariants

$$\mathrm{Sh} = \Phi(J_3) = \Phi(J_3^*/J_2^*). \tag{3.12}$$

The quantity J_3 in (3.12) varies from zero (plane shear) to the maximum value equal to $2^{1/3} \cdot 6^{-1/2} = 0.514$ (axisymmetric shear) when $J_1 = 0$ and $J_2 = 1$. Here, in view of Eqs. (3.10) and (3.11) the increment in mean Sherwood number in the entire interval of variation of J_3 is negligible and is less than a percent. This makes it possible to extend the hypothesis that the integral flow depends very weakly on the third invariant in (3.3) ($\Phi \approx$ const), as described for the solid particles in [6], to droplets and bubbles, and, apparently, makes it possible to use Eq. (3.10) for an approximate determination of the mean Sherwood number in the case of an arbitrary, pure shear flow past a spherical droplet.

4. MASS TRANSFER TO A CYLINDER IN ARBITRARY SHEAR FLOW

Consider a stationary convective diffusion to the surface of a fixed circular cylinder of radius a in an arbitrary, incompressible, homogeneous, linear shear flow. In the general (plane) case the velocity distribution in such a flow away from the cylinder can be described in the Cartesian coordinate system x_1, x_2 in the following form

$$r \to \infty,\ v_i = G_{ij}x_j + o(1);\ G_{ij}\delta_{ij} = 0\ (i,\ j = 1,2),$$
$$\begin{Vmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{Vmatrix} = \begin{Vmatrix} E_{11} & E_{12} \\ E_{12} & -E_{11} \end{Vmatrix} + \begin{Vmatrix} 0 & -\Omega \\ \Omega & 0 \end{Vmatrix}, \tag{4.1}$$
$$E_{11} = G_{11} = -G_{22},\ E_{12} = E_{21} = (1/2)(G_{12} + G_{21}),\ \Omega = (1/2)(G_{21} - G_{12}).$$

Here, as before, v_i and G_{ij} are nondimensional velocity components and shear tensor whose normalization is described below. For clarity the matrix of the shear stress coefficients $||G_{ij}||$ in (4.1) is written as a sum of symmetric $||E_{ij}||$ and antisymmetric $||\Omega_{ij}||$ quantities which correspond to pure deformation and pure rotation of the flow at infinity. In the general case the shear stress tensor is determined by specifying three independent quantities E_{11}, E_{12}, and Ω.

In the Stokes approximation the solution to the hydrodynamic problem on the velocity distribution, with boundary conditions at infinity (4.1) and no-slip condition at the surface of the cylinder $r = 1$, $v = 0$, can be obtained using results of [15] which leads to the following expression for the stream function:

$$\psi = \frac{1}{2}E_{11}\left(r - \frac{1}{r}\right)^2 \sin 2\theta - \frac{1}{2}E_{12}\left(r - \frac{1}{r}\right)^2 \cos 2\theta -$$
$$-\frac{1}{2}\Omega(r^2 - 1 - 2\ln r) = \frac{1}{2}\bar{E}\left(r - \frac{1}{r}\right)^2 \sin 2\bar{\theta} - \frac{1}{2}\Omega(r^2 - 1 - 2\ln r),\quad v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta},\quad v_\theta = -\frac{\partial\psi}{\partial r}, \tag{4.2}$$
$$\bar{E} = (E_{11}^2 + E_{12}^2)^{1/2},\quad \bar{\theta} = \theta + \Delta\theta\quad \left(\frac{E_{11}}{\bar{E}} = \cos(2\Delta\theta),\ \frac{E_{12}}{\bar{E}} = -\sin(2\Delta\theta)\right).$$

Here the coordinate system r, $\bar{\theta}$ ($r = \sqrt{x_1^2 + x_2^2}$), obtained from the initial system by rotation through an angle $\Delta\theta$, is fixed to the principal axes of the symmetric tensor E (in principal axes the tensor E is reduced to diagonal form with elements $\bar{E}$ and $-\bar{E}$). The quantity $\bar{E}$ determines the strength of pure deformation of the elasto-compressible motion of the fluid along the major axes and the parameter Ω corresponds to the angular speed (rigid body motion) of the flow at infinity. Simple shear is given by $E_{11} = 0$, $E_{12} = -\Omega = G_{12}/2$, $\bar{E} = |E_{12}|$ in Eqs. (4.1) and (4.2).

The structure of stream lines ψ = const (4.2) significantly depends on the ratio of parameters $\bar{E}$ and Ω. Without any loss of generality we assume $\bar{E} = 1$ in Eqs. (4.2) which corresponds to the choice of a length scale for shear $J_2/\sqrt{2} = (E_{11}^{*2} + E_{12}^{*2})^{1/2}$ in writing nondimensional equations (4.1).

The stream function (4.2) has the following limiting characteristics:

near the cylinder surface

$$r \to 1,\ \psi \approx (r-1)^2(2\sin 2\bar{\theta} - \Omega)\quad (\bar{E} = 1), \tag{4.3}$$

at infinity

$$r \to \infty,\ \psi \approx (1/2)r^2(\sin 2\bar{\theta} - \Omega)\quad (\bar{E} = 1). \tag{4.4}$$

We shall now restrict the study to the following range of angular velocity of the flow at infinity: $0 \leqslant |\Omega| \leqslant 1$. In this case, it follows from Eqs. (4.2)-(4.4) that all streamlines are open and there are four stagnation points on the surface of the cylinder:

$$\bar{\theta}_1 = \alpha,\quad \bar{\theta}_2 = \frac{\pi}{2} - \alpha,\quad \bar{\theta}_3 = \pi + \alpha,\quad \bar{\theta}_4 = \frac{3}{2}\pi - \alpha,\quad \alpha = \frac{1}{2}\arcsin\frac{\Omega}{2}, \tag{4.5}$$

where $\bar{\theta}_2$ and $\bar{\theta}_4$ correspond to inflow trajectories and $\bar{\theta}_1$ and $\bar{\theta}_3$ denote outflow trajectories. An increase in angular speed at infinity from zero to unity shifts the trailing edge stagnation point $\bar{\theta}_1$ counterclockwise by 15° (Fig. 3). Figure 4a, b shows streamlines corresponding to $\Omega = 0$ (pure shear) and $\Omega = 1$ (simple shear).

The concentration distribution and the local diffusive flow on the surface of the cylinder are given by Eqs. (2.4), (2.5), and (2.7) where

$$\xi = r - 1,\quad \eta = \bar{\theta},\quad n = 2,\quad g^0 = g^0_{\xi\xi} = 1,\quad f = 2\sin\theta - \Omega, \tag{4.6}$$

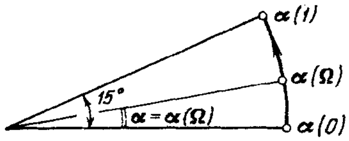

Fig. 3

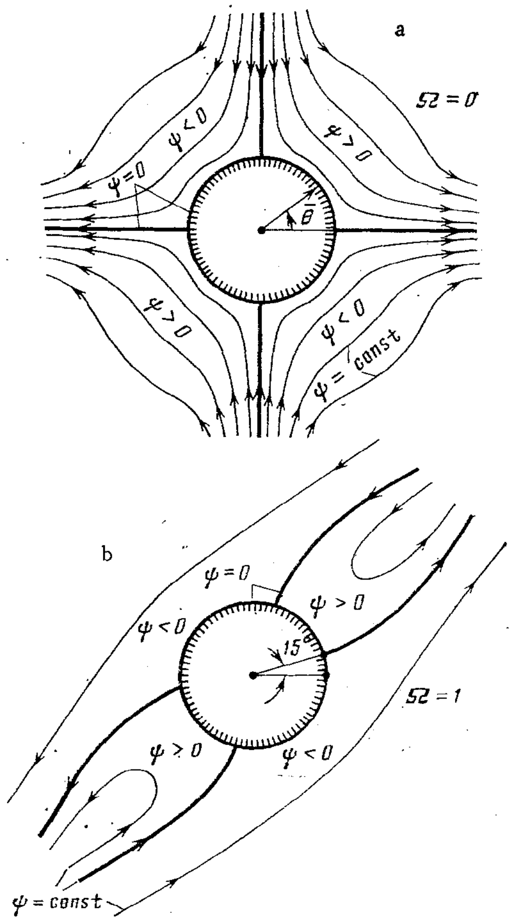

Fig. 4

$$\bar{\theta}^- = \eta^- = \begin{cases} \frac{\pi}{2} - \alpha & \text{for } \alpha \leqslant \bar{\theta} \leqslant \pi + \alpha, \\ \frac{3}{2}\pi - \alpha & \text{for } \pi + \alpha \leqslant \bar{\theta} \leqslant 2\pi + \alpha. \end{cases}$$

Using Eqs. (2.6), (2.7), and (4.6) and taking into account the equalities

$$t(\bar{\theta}_2, \bar{\theta}_1) = t(\bar{\theta}_4, \bar{\theta}_3) = 2E\left(\tfrac{1}{2}\sqrt{2-\Omega}\right) - \left(1 + \tfrac{\Omega}{2}\right) K\left(\tfrac{1}{2}\sqrt{2-\Omega}\right),$$
$$t(\bar{\theta}_3, \bar{\theta}_2) = t(\bar{\theta}_5, \bar{\theta}_4) = 2E\left(\tfrac{1}{2}\sqrt{2+\Omega}\right) - \left(1 - \tfrac{\Omega}{2}\right) K\left(\tfrac{1}{2}\sqrt{2+\Omega}\right),$$

where K and E are complete elliptic integrals of the first and second kind respectively ($\bar{\theta}_5 \equiv \bar{\theta}_1$), we obtain mean Sherwood number on the surface of the cylinder

$$\mathrm{Sh} = \frac{3^{4/3}}{\pi\Gamma(1/3)} \left\{\left[2E\left(\tfrac{1}{2}\sqrt{2-\Omega}\right) - \left(1+\tfrac{\Omega}{2}\right)K\left(\tfrac{1}{2}\sqrt{2-\Omega}\right)\right]^{2/3} + \left[2E\left(\tfrac{1}{2}\sqrt{2+\Omega}\right) - \left(1-\tfrac{\Omega}{2}\right)K\left(\tfrac{1}{2}\sqrt{2+\Omega}\right)\right]^{2/3}\right\} \mathrm{Pe}^{1/3}, \tag{4.7}$$
$$I = 2\pi\,\mathrm{Sh}, \quad \mathrm{Pe} = a^2\bar{E}^* D^{-1}$$
$$(0 \leqslant |\Omega| \leqslant 1).$$

In writing Eq. (4.7) it was assumed that in plane problems integral flow is usually computed per unit length of the cylinder which corresponds to the value $\Lambda = 1$ ($\lambda = x_3$) in Eq. (2.6).

In certain cases of pure shear ($\Omega = 0$) and simple ($|\Omega| = 1$) linear shear flow past a cylinder we obtain from Eq. (4.7)

$$\mathrm{Sh} = 0.920\,\mathrm{Pe}^{1/3} \tag{4.8}$$

$$(\Omega = 0);\ \mathrm{Sh} = 0.908\ \mathrm{Pe}^{1/3} \quad (|\Omega| = 1).$$

It follows from Eq. (4.7) that an increase in the absolute value of the angular velocity of rotation of the shear flow Ω leads to a decrease in mass and heat transfer of the cylinder with the surrounding medium. As seen from Eq. (4.8), the mean Sherwood number changes very little in the given range (relative increment in mean Sherwood number with increase in $|\Omega|$ from zero to unity is only 1.3%). This last condition allows an approximate computation of mean Sherwood number corresponding to an arbitrary shear flow past a fixed circular cylinder, and use of the first Eq. (4.8) (instead of the exact Eq. (4.7)) in the entire range $0 \leqslant |\Omega| \leqslant 1$.

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